

# CONCORDANCE OF SEIFERT SURFACES

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**ABSTRACT.** This paper proves that every non-disk Seifert surface  $F$  for a knot  $K$  in  $S^3$  is smoothly concordant to a Seifert surface  $F'$  for a hyperbolic knot  $K'$  of arbitrarily large volume. This gives a new and simpler proof of the result of Friedl and of Kawauchi that every knot is  $S$ -equivalent to a hyperbolic knot of arbitrarily large volume. The construction also gives a new and simpler proof of the result of Silver and Whitten and of Kawauchi that for every knot  $K$  there is a hyperbolic knot  $K'$  of arbitrarily large volume and a map of pairs  $f : (S^3, K') \rightarrow (S^3, K)$  which induces an epimorphism on the knot groups. An example is given which shows that knot Floer homology and the Jones polynomial are not invariants of Seifert surface concordance. The paper also proves that a set of finite volume hyperbolic 3-manifolds with unbounded Haken numbers has unbounded volumes.

## 1. INTRODUCTION

In what follows the smooth category will always be assumed. This paper concerns two equivalence relations on knots in  $S^3$ , concordance and  $S$ -equivalence. Knots  $K$  and  $K'$  are *concordant* if there is a properly embedded annulus  $A$  in  $S^3 \times [0, 1]$  with  $A \cap (S^3 \times \{0\}) = K$  and  $A \cap (S^3 \times \{1\}) = K'$ . Knots  $K$  and  $K'$  are  *$S$ -equivalent* if they have Seifert surfaces  $F$  and  $F'$  with associated Seifert matrices which are equivalent under integral congruence and elementary expansions and contractions [29].

Concordant knots need not be  $S$ -equivalent, e.g. the trivial knot and a slice knot with non-trivial Alexander polynomial, such as the stevedore's knot  $6_1$ .  $S$ -equivalent knots need not be concordant; Kearton [11] has shown that every algebraically slice knot is  $S$ -equivalent to a slice knot, but by Casson and Gordon [1] there are algebraically slice knots which are not slice knots.

The author [20] proved that every knot is concordant to a hyperbolic knot, generalizing the result of Kirby and Lickorish [13] that every knot is concordant to a prime knot. Friedl [3] and Kawauchi [7, 8, 10] have given proofs that every knot is  $S$ -equivalent to a hyperbolic knot, generalizing the result of Kearton [11] that every knot is  $S$ -equivalent to a prime knot.

Friedl showed that two knots are  $S$ -equivalent if and only if they have isometric Blanchfield pairings. He then noted that by combining two results of Kawauchi's imitation theory of knots (Theorem 1.1 of [8] and Properties I and V of [7]) one gets that for every knot  $K$  there is a hyperbolic knot  $K'$  of arbitrarily large volume and a map  $f : (S^3, K') \rightarrow (S^3, K)$  which induces isomorphisms on every quotient

of the knot groups by their derived subgroups. Friedl then showed that this result implies that the Blanchfield pairings are isometric. He also added a note in proof that one can combine the existence of such knots and maps with another result of Kawauchi (Theorem 2.2 of [10]) to show  $S$ -equivalence.

It is natural to ask whether for every knot  $K$  there is a hyperbolic knot  $K'$  of arbitrarily large volume to which  $K$  is both  $S$ -equivalent and concordant. It turns out that an affirmative answer is implicit in Kawauchi's construction in [8]. One can see the concordance by looking at Figure 7 of that paper for the time interval  $0 \leq t \leq 1$ .

Silver and Whitten [26] proved that given any triple  $(G, \mu, \lambda)$  where  $G$  is a knot group and  $(\mu, \lambda)$  is a meridian-longitude pair for  $G$  there are infinitely many triples  $(\tilde{G}, \tilde{\mu}, \tilde{\lambda})$  where  $\tilde{G}$  is the group of a prime knot and there is an epimorphism  $\phi : (\tilde{G}, \tilde{\mu}, \tilde{\lambda}) \rightarrow (G, \mu, \lambda)$ . In [27] they strengthened this to the  $\tilde{G}$  being the groups of hyperbolic knots of arbitrarily large volume. In a note in proof they added the comment that Kawauchi informed them that many of the results in the paper can be found in [8] and [9].

The constructions of Kawauchi and of Silver and Whitten mentioned above are rather intricate and the proofs somewhat lengthy. In the present paper the author gives a simpler and shorter construction and proof. Recall that a *Seifert surface*  $F$  for a knot  $K$  in  $S^3$  is a compact, orientable surface with boundary  $K$ . A Seifert surface  $F'$  for a knot  $K'$  will be said to be concordant to  $F$  if there is an embedding of  $F \times [0, 1]$  in  $S^3 \times [0, 1]$  which meets  $S^3 \times \{0\}$  in  $F$  and  $S^3 \times \{1\}$  in  $F'$ . Then  $K$  and  $K'$  are clearly concordant, and since  $F$  and  $F'$  have congruent Seifert matrices these knots are also  $S$ -equivalent. The main result of this paper is the following.

**Theorem 1.1.** *Let  $F$  be a Seifert surface for a knot  $K$  in  $S^3$ . Assume that  $F$  is not a disk. Then  $F$  is concordant to a Seifert surface  $F'$  for a knot  $K'$  such that*

- (a)  $K'$  is hyperbolic,
- (b)  $S^3 - K'$  has arbitrarily large Haken number,
- (c)  $S^3 - K'$  has arbitrarily large volume, and
- (d) *there is a map of pairs  $f : (S^3, K') \rightarrow (S^3, K)$  which induces an epimorphism  $f_* : \pi_1(S^3 - K') \rightarrow \pi_1(S^3 - K)$ .*

The paper is organized as follows. Section 2 reviews some basic material. Section 3 proves that every non-disk Seifert surface for a knot can be put in a certain standard position. Section 4 uses standard position to prove (a). Section 5 proves (b). The proof that (b) implies (c) follows from a more general result, that a set of finite volume hyperbolic 3-manifolds with unbounded Haken numbers has unbounded volumes. This fact appears to be “quasi-known” but the author does not know a precise reference in the literature and so gives a simple proof in the Appendix. Section 6 proves (d). Section 7 considers the question of whether Seifert surface concordance implies the invariance of more than just the union of the sets of invariants of concordance and  $S$ -equivalence. It rules out two candidates. It shows by example that although the Alexander polynomial is an invariant of Seifert surface concordance its categorification, knot Floer homology, is not. The same example

also shows that neither is the Jones polynomial.

## 2. PRELIMINARIES

As general references on knot theory and on 3-manifolds see [15] and [5]. A compact, connected, orientable 3-manifold  $M$  will be called *excellent* if it is irreducible, boundary-irreducible, anannular, atoroidal, and is not a 3-ball. By Thurston's uniformization theorem (see e.g. [17]) such manifolds are hyperbolic.

The following standard technical result will be used to build more complicated hyperbolic 3-manifolds out of simpler pieces. A proof can be found in [20].

**Lemma 2.1** (Gluing Lemma). *Let  $X$  be a compact, connected 3-manifold. Suppose  $F$  is a compact, properly embedded, two-sided 2-manifold in  $X$ . It is not assumed that  $F$  is connected. Let  $Y$  be the 3-manifold obtained by splitting  $X$  along  $F$ . Denote by  $F_1$  and  $F_2$  the two copies of  $F$  in  $\partial Y$  which are identified to obtain  $X$ . If each component of  $Y$  is excellent,  $F_1 \cup F_2$  and the closure of  $Y - (F_1 \cup F_2)$  are incompressible in  $Y$ , and each component of  $F_1 \cup F_2$  has negative Euler characteristic, then  $X$  is excellent.*

An  $n$ -tangle is the disjoint union  $\lambda = \lambda_1 \cup \dots \cup \lambda_n$  of properly embedded arcs in a 3-ball  $B$ . This is sometimes denoted by the pair  $(B, \lambda)$ . It will always be assumed that  $n \geq 2$ . When the specific number  $n$  of arcs is not at issue or is clear from the context  $\lambda$  will just be called a *tangle*.

This paper will assume that  $B$  is given a product structure of the form  $[a, b] \times [c, d] \times [e, f]$  with each component  $\lambda_i$  of the tangle  $(B, \lambda)$  joining a point of  $(a, b) \times (c, d) \times \{f\}$  to a point of  $(a, b) \times (c, d) \times \{e\}$ . This is done so that one may compose tangles. The product of the tangles  $(B, \lambda)$  and  $(B, \mu)$  will be obtained by setting  $(B, \lambda)$  on top of  $(B, \mu)$  so that the lower endpoint of each  $\lambda_i$  equals the upper endpoint of each  $\mu_i$ .

The *exterior* of a submanifold of a 3-manifold is the closure of the complement of a regular neighborhood of the submanifold. A knot or tangle will be called *excellent* if its exterior is excellent. In this case by a slight abuse of language the knot or tangle will be called hyperbolic.

## 3. STANDARD POSITION FOR SEIFERT SURFACES

Let  $F$  be a non-disk Seifert surface for a knot  $K$ . This section defines a way of presenting  $F$  by a diagram in the plane, standard position. This presentation will be used to build concordances.

The first step is to define an analogue for certain graphs of a plat presentation of a knot.

Recall the idea of a plat presentation for a knot or link. Regard  $S^3$  as the union of a 3-ball  $B^+$ , a copy  $B^0$  of  $S^2 \times [-1, 1]$ , and a 3-ball  $B^-$  with  $B^+ \cap B^0 = S^2 \times \{1\}$

and  $B^- \cap B^0 = S^2 \times \{-1\}$ . Choose  $M$  unknotted, unlinked, properly embedded arcs in each of  $B^+$  and  $B^-$ . Arrange them so that the projections  $p^\pm : B^\pm \rightarrow D^\pm$  onto equatorial disks have no crossings and are unnested. Let  $p^0 : B^0 \rightarrow A$  be projection of  $B^0$  onto an annulus  $A$ . Join the endpoints of the arcs in  $B^+$  to the endpoints of the arcs in  $B^-$  by arcs in  $B^0$  such that projection on  $A$  is given by a braid  $\beta$ . In this paper the braid will be chosen so that it is an element of the braid group of the plane, not the braid group of the sphere. The knot projections will be drawn in  $\mathbb{R}^2$  with regions  $U$ ,  $C$  and  $L$  representing the projections of regions near the knot in  $B^+$ ,  $B^0$ , and  $B^-$ , respectively. The map  $p : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  denotes this local projection. The vertical coordinate in  $\mathbb{R}^2$  is denoted by  $y$ . The braid is represented by a box labelled  $\beta$ . The arcs in  $B^+$  and  $B^-$  are represented by arcs on the top and bottom of the box, respectively.

Now suppose that  $M \geq 2$ . Choose  $g$  such that  $2 \leq 2g \leq M$ , and let  $m = M - 2g$ . Replace  $2g$  adjacent arcs in  $L$  with the cone  $X$  on their boundary points having vertex  $v$  in the interior of  $L$ ; require that  $X$  be disjoint from the remaining arcs in  $L$ . This gives a 1-complex in  $\mathbb{R}^3$  consisting of a wedge of  $2g$  circles with possibly some additional simple closed curve components. Restrict attention to those  $\beta$  for which there are no such additional components. By changing  $\beta$  one may obtain an isotopic embedding of  $W$  such that  $X$  is to the left of the arcs in  $L$ . This will be called a *plat presentation of a wedge of  $2g$  circles*. See Figure 1.

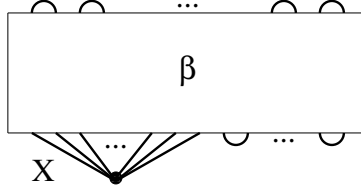


FIGURE 1. Plat presentation of a wedge of circles

One may add words to the top and bottom of  $\beta$  to obtain an isotopic embedding of  $W$  so that the arcs in each of  $U$  and  $L$  are concentrically nested. This will be called a *standard presentation of a wedge of  $2g$  circles*. See Figure 2.

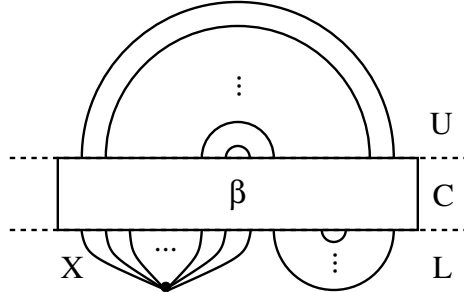


FIGURE 2. Standard presentation of a wedge of circles

Now take regular neighborhoods in  $U \cup L$  of the arcs and  $X$  and widen the arcs in  $\beta$  to bands which cross in the same manner as the arcs. One gets a *standard presentation of a surface with boundary*. The special case of interest is that of an orientable surface with connected boundary.

**Lemma 3.1.** *Every non-disk Seifert surface for a knot in  $S^3$  has a standard presentation.*

*Proof.* Let  $F$  be a genus  $g \geq 1$  Seifert surface for a knot. Choose a wedge of circles  $W$  in the interior of  $F$  such that the boundary of a regular neighborhood of  $W$  in  $F$  is parallel in  $F$  to  $K$ . Let  $v$  be the vertex of  $W$ . The number of circles is  $2g$ , where  $g$  is the genus of  $F$ . Isotop  $F$  so that the projection of  $W$  onto  $\mathbb{R}^2$  has only transverse double point singularities. In particular  $p^{-1}(p(v))$  consists of a single point. Isotop  $F$  so that  $p(v)$  has  $y$  coordinate less than or equal to that of any other point of  $p(W)$ . We may assume that for each edge of  $W$  all the critical values of the function  $y \circ p$  are local maxima and minima. Isotop  $F$  so as to move all the minima into  $L$  and all the maxima into  $U$ . One now has a plat presentation of  $W$ .

The only obstruction to widening  $W$  into a plat presentation for  $F$  is that some of the bands may be twisted. Since  $F$  is orientable the twisting in each band consists of a number of full twists. Each full twist is isotopic to a curl, as illustrated in Figure 3. Isotop the local maxima and local minima of the curls into  $U$  and  $L$ , respectively. The isotopy from plat to standard presentation then preserves the fact that the bands are untwisted.  $\square$

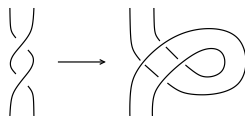


FIGURE 3. Replacing a twist by a curl

#### 4. CONSTRUCTING A CONCORDANCE

The basic idea is to construct a concordance from  $W$  to some  $W'$  and thereby get a concordance from  $F$  to  $F'$ . However, the obvious surface one gets from a projection of  $W'$  might not be concordant to  $F$ . As an example of this phenomenon let  $P$  be a zero crossing projection of the trivial knot and  $P'$  a one crossing projection of the trivial knot. Let  $A$  and  $A'$  be annuli obtained by widening  $P$  and  $P'$  into bands. These annuli cannot be concordant because the components of their boundaries have different linking numbers. The way to fix this is to require that any isotopies of a diagram be *regular*, i.e. they are composed of Reidemeister moves of types II and III together with planar isotopies.

**Lemma 4.1.** *Given a standard presentation of a non-disk Seifert surface  $F$ , there is a concordance of  $F$  with a Seifert surface  $F'$  such that  $S^3 - K'$  is hyperbolic, where  $K' = \partial F'$ .*

*Proof.* By the gluing lemma it will be sufficient to show that  $S^3 - F'$  is excellent.

Let  $W$  be the wedge of circles of which  $F$  is a regular neighborhood. Choose a 3-ball  $E^+$  in  $U$  and a 3-ball  $E^-$  in  $L$  as shown in Figure 4.  $E^+$  meets  $C$  in a disk and meets  $W$  in  $M$  straight arcs; these arcs meet  $C$  in the leftmost  $M$  points of  $W \cap U \cap C$ .  $E^-$  meets  $C$  in a disk and meets  $W$  in all but the first endpoint of  $X$  and in the leftmost  $M - 2g$  endpoints of the arcs in  $(W \cap L) - X$ .

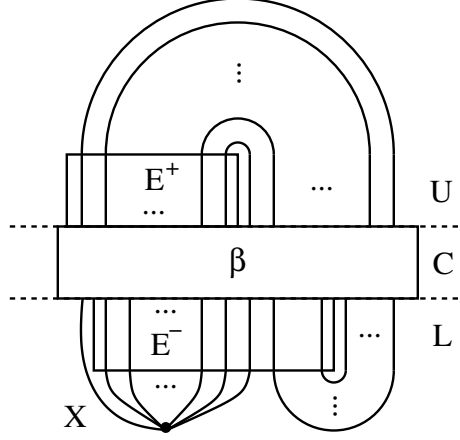


FIGURE 4.  $W$  before surgery

The trivial tangles in  $E^+$  and in  $E^-$  are then replaced by concordant hyperbolic tangles  $\alpha$  and  $\gamma$ , respectively. The process starts with a hyperbolic  $n$ -tangle constructed in [20]. Figure 5 shows the  $n = 4$  case.

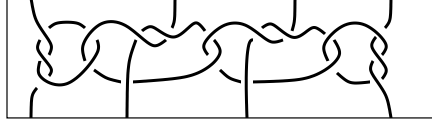


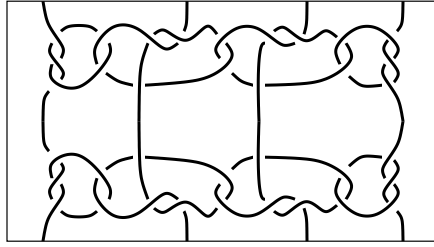
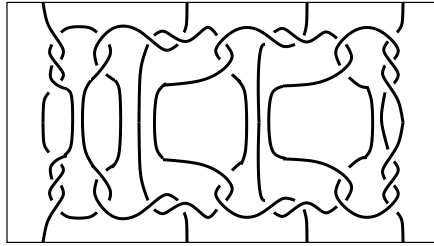
FIGURE 5. A hyperbolic  $n$ -tangle,  $n = 4$

This tangle is then composed with its mirror image as in Figure 6.

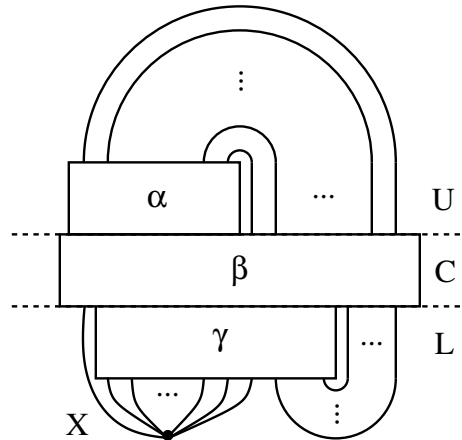
A concordance to the trivial tangle is then constructed by attaching 1-handles, performing Type II Reidemeister moves, and then attaching 2-handles. See Figure 7 for the first stage.

For future reference the general version of this result is stated below.

**Lemma 4.2.** *Let  $D$  be a disk. Let  $\tau = \tau_1 \cup \dots \cup \tau_n$  be an  $n$ -tangle in  $D \times [0, 1]$  such that each  $\tau_i$  has one boundary point in  $\text{int}(D) \times \{0\}$  and the other in  $\text{int}(D) \times \{1\}$ . Let  $\delta(\tau)$  be the tangle in  $D \times [-1, 1]$  obtained by taking the union of  $\tau$  and its mirror image in  $D \times [-1, 0]$ . Then  $\delta(\tau)$  is concordant to the trivial tangle  $\varepsilon$ .  $\square$*

FIGURE 6. A ribbon  $n$ -tangle,  $n = 4$ FIGURE 7. A ribbon concordance,  $n = 4$ 

Continuing with the proof of Lemma 4.1 one next shows that  $S^3 - F'$  is hyperbolic. Figure 8 shows the new wedge of circles  $W'$ . The exterior of  $W' \cap U$  is homeomorphic to the exterior of the tangle  $\alpha$  and is therefore hyperbolic. The exterior of  $W' \cap L$  is homeomorphic to the exterior of the tangle  $\gamma$ . This can be seen as follows. Denote the arcs in  $X$  by  $\gamma_1, \dots, \gamma_k$ , numbered from left to right. Slide the lower endpoint of  $\gamma_2$  along  $\gamma_1$  and into  $C \cap L$ . Continue with  $\gamma_3$  through  $\gamma_{2g-1}$ . The complement in  $L$  of the new set of arcs is homeomorphic to the complement of  $\gamma$  and is therefore hyperbolic. Since the exterior of  $\beta$  in  $C$  is a product the result follows.  $\square$

FIGURE 8.  $W'$  after surgery.

## 5. RAISING THE HAKEN NUMBER AND THE VOLUME

**Lemma 5.1.** *Given a standard presentation of a non-disk Seifert surface  $F$ , and given an  $N > 0$  there is a concordance of  $F$  with a Seifert surface  $F'$  such that  $K'$  is hyperbolic and  $S^3 - K'$  has Haken number at least  $N$ .*

*Proof.* In the construction of the previous section replace each of the tangles  $\alpha$  and  $\gamma$  by  $N$  copies of itself stacked one on top of the other. This gives  $N$  disjoint copies of the exteriors of  $\alpha$  and  $\gamma$ . The boundaries of the exteriors of these tangles are all incompressible and non-parallel in  $S^3 - K'$ .  $\square$

**Lemma 5.2.** *Given a standard presentation of a non-disk Seifert surface  $F$ , and given a  $V > 0$  there is a concordance of  $F$  with a Seifert surface  $F'$  such that  $K'$  is hyperbolic and  $S^3 - K'$  has volume at least  $N$ .*

*Proof.* This follows from Proposition A.1 in the Appendix and Lemma 5.1.  $\square$

## 6. MAKING A MAP

**Lemma 6.1.** *Given a standard presentation of a non-disk Seifert surface  $F$ , there is a concordance of  $F$  with a Seifert surface  $F'$  satisfying (a), (b), (c), and (d).*

*Proof.* The proof follows from Lemmas 6.2 and 6.3 below by defining  $f$  to be the identity outside the tangles involved.  $\square$

A tangle  $(B^3, \tau_1 \cup \dots \cup \tau_n)$  is a *boundary tangle* if there are disjoint arcs  $\sigma_1, \dots, \sigma_n$  in  $\partial B^3$  such that  $\partial\sigma_i = \partial\tau_i$  and disjoint, compact, orientable surfaces  $G_i$  in  $B^3$  such that  $\partial G_i = \tau_i \cup \sigma_i$ . Let  $\tau = \tau_1 \cup \dots \cup \tau_n$  and  $G = G_1 \cup \dots \cup G_n$ .

Let  $\tau^* = \tau_1^* \cup \dots \cup \tau_n^*$  be a trivial tangle in  $B^3$  with  $\partial\tau_i^* = \partial\tau_i$  for all  $i$ . Choose disjoint disks  $G_i^*$  in  $B^3$  with  $\partial G_i^* = \partial G_i$ .

**Lemma 6.2.** *There is a map  $g : (B^3, \tau, B^3 - \tau) \rightarrow (B^3, \tau^*, B^3 - \tau^*)$  which is the identity on  $\partial B^3$  and a homeomorphism from  $\tau$  to  $\tau^*$ . In particular  $g$  induces an epimorphism  $\pi_1(B^3 - \tau) \rightarrow \pi_1(B^3 - \tau^*)$  which carries the meridians of  $\tau$  to the meridians of  $\tau^*$ .*

*Proof.* Let  $Y$  be the exterior of  $\tau$  in  $B^3$ . Let  $H_i = G_i \cap Y$ . Let  $N_i = H_i \times [-1, 1]$  be a regular neighborhood of  $H_i$  in  $Y$ . Let  $N$  be the union of the  $N_i$ . Let  $C$  be a collar on  $\partial Y$  whose intersection with each  $N_i$  has the form  $A_i \times [-1, 1]$ , where  $A_i$  is a collar on  $\partial H_i$  in  $H_i$ .

In a similar fashion let  $Y^*$  be the exterior of  $\tau^*$  in  $B^3$ , let  $H_i^* = G_i^* \cap Y^*$ ,  $N_i^* = H_i^* \times [-1, 1]$ ,  $N^* = \cup N_i^*$ ,  $C^* = C$ , and  $A_i^* = A_i$ .

For each  $t \in [-1, 1]$  define a map from  $N_i$  to  $N_i^*$  by crushing  $(H_i - C) \times \{t\}$  to a point in  $H_i^*$ . This defines the restriction of  $g$  to  $H_i \times [-1, 1]$ .

One next defines  $g$  on the closure  $W$  of the complement of  $N$  in  $B^3$  by crushing  $W - C$  to a point. This gives a quotient map onto a 3-ball which may be identified with the closure of  $B^3 - N^*$ .



Putting the two quotient maps together gives a quotient map from  $Y$  to  $Y^*$  which extends to a map  $g : B^3 \rightarrow B^3$  with the required properties.  $\square$

The existence of hyperbolic boundary  $n$ -tangles was proven by Cochran and Orr of [2, Lemma 7.3 on pp. 519-520] in a more abstract general setting. In keeping with the desire to make the constructions in this paper as explicit as possible their procedure is implemented in the following specific construction.

**Lemma 6.3** (Cochran-Orr). *Hyperbolic boundary  $n$ -tangles exist.*

*Proof.* First the hyperbolic  $n$ -tangle  $\lambda$  is placed in a larger box and connected by straight arcs going to the boundary of the larger box as in the first diagram in Figure 9. Then the endpoints of every second arc are slid onto the arc preceding it as in the second diagram. Then the bottom endpoints of each resulting graph are slid across the front of the larger box onto the top arc as in the third diagram. This does not change the homeomorphism type of the exterior of the graph.

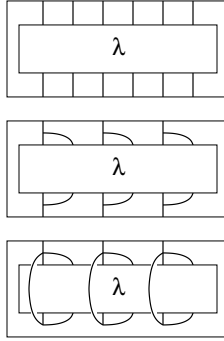


FIGURE 9. Sliding endpoints to obtain a graph

Then one constructs the  $2n$ -tangle  $\Delta(\lambda)$  by replacing each arc of  $\lambda$  by two parallel copies as in Figure 10.

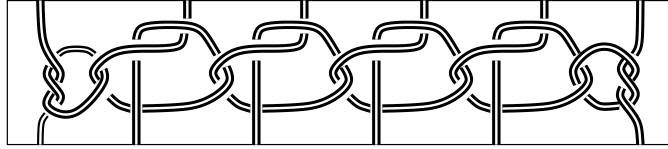


FIGURE 10. Double  $\Delta(\lambda_n)$  of  $\lambda_n$ ,  $n = 6$

Next one modifies the last diagram of Figure 9 in the following ways to obtain Figure 11. First, one replaces  $\lambda$  in the inner box by  $\Delta(\lambda)$ . Second, one widens the graphs which join the inner box to the boundary of the outer box to obtain surfaces whose unions with the bands inside the inner box are punctured tori. Note the half-twists inserted into the lower portion of Figure 11 to achieve this. By the

gluing lemma the exterior of this new tangle is hyperbolic since it is obtained from a 3-manifold homeomorphic to the exterior of  $\lambda$  by identifying pairs of incompressible once-punctured tori in its boundary.

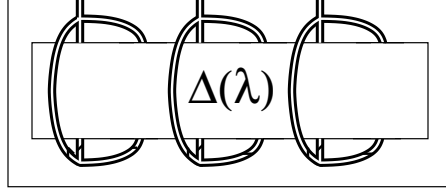


FIGURE 11. A hyperbolic boundary tangle

Finally one slides the left endpoints of each arc across the front of the larger box (dotted lines) to obtain the final hyperbolic boundary tangle as in Figure 12. The union of this tangle with its mirror image is still a boundary tangle and the rest of the proof proceeds as before.  $\square$

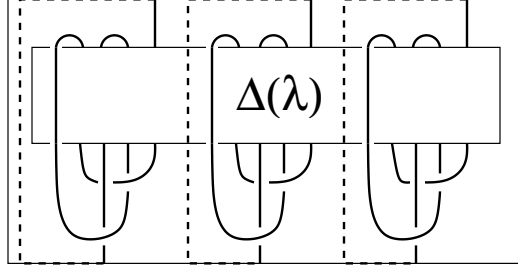


FIGURE 12. The hyperbolic boundary tangle ready for use

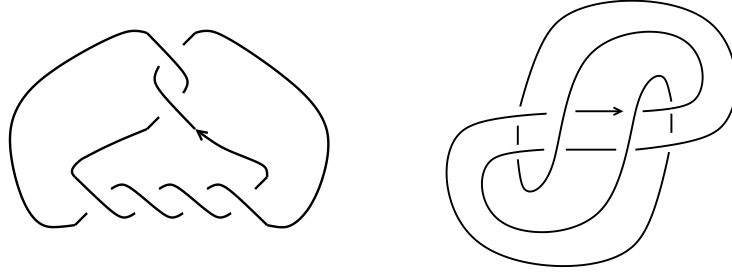
## 7. NON-INVARIANCE OF KNOT FLOER HOMOLOGY AND THE JONES POLYNOMIAL

This section gives an example of a knot  $J$  with Seifert surface  $F$  and a knot  $J'$  with Seifert surface  $F'$  such that  $F$  and  $F'$  are concordant, but  $J$  and  $J'$  have different knot Floer homology and different Jones polynomial.

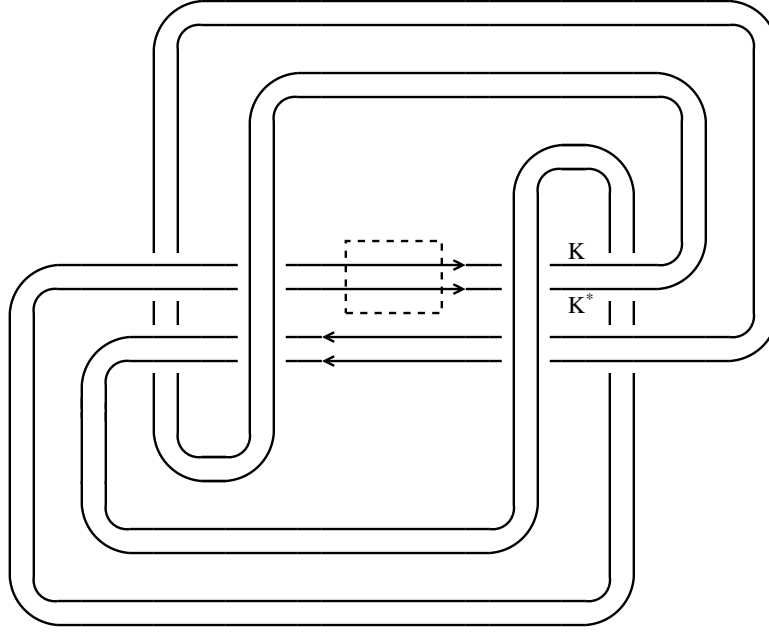
$J$  is the trefoil knot and  $F$  a genus one Seifert surface.  $J'$  is a certain twisted double of a copy  $K$  of the stevedore's knot  $6_1$  and  $F'$  is a genus one surface contained in a solid torus  $V$  whose core is  $K$ .

Figure 13 shows two projections of  $K$ . The second will be used since it has zero writhe, which simplifies some calculations, and it more clearly displays the fact that  $K$  is a ribbon knot. See [16] for an isotopy between them.

Figure 14 shows an annulus  $A$  embedded in  $S^3$  with one boundary component being  $K$ . The orientations of  $K$  and the other boundary component  $K^*$  are chosen so that the two curves are homologous in  $A$ . From the diagram one computes that

FIGURE 13. Two projections of the stevedore's knot  $6_1$ .

the linking number  $\text{lk}(K, K^*) = 0$ . Let  $\tilde{K} = K \cup K^*$ .

FIGURE 14. A 2-strand cable link  $\tilde{K} = K \cup K^*$  of  $K = 6_1$ .

The knot  $J'$  is obtained by reversing the orientation on  $K^*$  to get a new oriented link  $\hat{K} = K \cup (-K^*)$  and then replacing the trivial tangle in the box in Figure 14 by the tangle in Figure 15.



FIGURE 15.

This results in the annulus being replaced by a genus one surface  $F'$ . A saddle move on the new diagram followed by a pair of 2-handle additions shows that  $F$  and  $F'$  are concordant.

By a result of Ni [22] knot Floer homology detects fibered knots.  $J$  is fibered. If  $J'$  were fibered then its companion  $K$  would also be fibered (Proposition 9.11 of [18]), but the Alexander polynomial  $2t^{-1} - 5 + 2t$  of  $K$  is not monic, and so  $K$  is not fibered (see e.g. [24, Corollary 10.8]), thus  $J$  is not fibered, and so the knot Floer homologies of  $J$  and  $J'$  must be different.

The Jones polynomial of the oriented knot  $J$  is  $V(J) = t^{-4} + t^{-3} + t^{-1}$  (e.g. [15, p. 26]). The computation of the Jones polynomial  $V(J')$  uses the following result of King [12]. For a general oriented knot  $K$  let  $F_K(a, z)$  be the two variable Kauffman polynomial [6]. Let  $\tilde{K}$  be as described above (but with no restriction on  $f = \text{lk}(K, K^*)$ ). Then the Jones polynomial  $V(\tilde{K})$  satisfies the following equation.

$$t^f(1 + t + t^{-1})F_K(it^{-2}, i(t - t^{-1})) = -(t^{1/2} + t^{-1/2})V(\tilde{K}) - t^{3f}$$

Returning to the case at hand we have [15, p. 183])

$$\begin{aligned} F_K(a, z) = & (-a^{-2} + a^2 + a^4) + (2a + 2a^3)z + (a^{-2} + 4a^2 - 3a^4)z^2 + \\ & (a^{-1} - 2a - 3a^3)z^3 + (1 + 2a^2 + a^4)z^4 + (a + a^3)z^5 \end{aligned}$$

One computes that

$$\begin{aligned} F(it^{-2}, i(t - t^{-1})) = & t^{-12}(t^{18} - t^{17} + 2t^{15} - 3t^{14} + 4t^{12} - 4t^{11} + 4t^9 - 3t^8 + 3t^6 \\ & - 2t^5 - t^4 + 2t^3 - t^2 - t + 1). \end{aligned}$$

Using the fact that  $f = 0$  one then computes that

$$V(\tilde{K}) = -t^{-25/2}(t^{19} - t^{18} + t^{17} - t^{15} + t^{13} + t^9 + t^6 - t^5 - t + 1).$$

$$V(\hat{K}) = t^{-3f}V(\tilde{K}) \text{ (e.g. [15, p. 26]). So in this case one has } V(\hat{K}) = V(\tilde{K}).$$

Recall the skein relation for the Jones polynomial (p. 28 of [15]).

$$t^{-1}V(L_+) - tV(L_-) + (t^{-1/2} - t^{1/2})V(L_0) = 0$$

where  $L_+$ ,  $L_-$ , and  $L_0$  are as in Figure 16.



FIGURE 16. Skein relation for the Jones polynomial.

Applying this to one of the crossings on the right hand side of the tangle in Figure 15 one gets

$$t^{-1}V(\text{diagram}) - tV(\text{diagram}) + (t^{-1/2} - t^{1/2})V(\text{diagram}) = 0.$$

The diagram on the left gives  $J'$  and the middle diagram gives the trivial knot so one has

$$V(J') = t^2 + (t^{3/2} - t^{1/2})V(\text{diagram}).$$

Applying the skein relation to the diagram on the right gives

$$t^{-1}V(\text{diagram}) - tV(\text{diagram}) + (t^{-1/2} - t^{1/2})V(\text{diagram}) = 0.$$

The diagram in the middle is  $\widehat{K}$ , and the diagram on the right is the trivial knot, so one then has

$$V(\text{diagram}) = t^2 + V(\widehat{K}) + t^{3/2} - t^{1/2}.$$

Substituting this in the formula for  $V(J')$  gives

$$V(J') = t^3 - t^2 + t + (t^{7/2} - t^{5/2})V(\widehat{K}).$$

Inserting  $V(\widehat{K}) = V(\widetilde{K})$  gives

$$V(J') = -t^{-10}(t^{20} - 2t^{19} + 2t^{18} - t^{17} - t^{16} + t^{15} + t^{14} - 2t^{13} + t^{12} - t^{11} + t^{10} - t^9 + t^7 - 2t^6 + t^5 - t^2 + 2t - 1)$$

## APPENDIX A. PUMPING UP THE VOLUME

There are several results in the literature to the effect that a topologically complicated hyperbolic 3-manifold has high volume. See for example [14, 23, 25].

One measure of the complexity of a compact 3-manifold  $M$  is the *Haken number*  $h(M)$  [4, 5], the maximum number of compact, connected, properly embedded, incompressible, boundary incompressible, pairwise non-parallel surfaces in  $M$ . Call the union of such a maximal collection of surfaces a *Haken system* for  $M$ .

**Theorem A.1.** *Let  $M_n$  be a sequence of compact, connected, orientable 3-manifolds with complete, finite volume, hyperbolic interiors  $N_n$ .*

$$\text{If } \lim_{n \rightarrow \infty} h(M_n) = \infty, \text{ then } \lim_{n \rightarrow \infty} \text{Vol}(N_n) = \infty.$$

*Proof.* If not, then by passing to a subsequence we may assume that  $\text{Vol}(M_n)$  is bounded above by a positive constant  $V$ . It then follows from Jørgensen's Theorem [28, Theorem 5.12.1, p. 119] that there is a finite set  $\{X_1, \dots, X_r\}$  of compact, connected, orientable 3-manifolds with finite volume, hyperbolic interiors such that for each  $M_n$  there is an  $X_j$  such that  $M_n$  is homeomorphic to the result of Dehn filling along some of the components of  $\partial X_j$ . Let  $H$  be the maximum of the  $h(X_j)$ . Choose an  $n$  such that  $h(M_n) > H$ . By the following lemma, which is stated in greater generality than needed here, we have that  $h(M_n) \leq H$ .  $\square$

**Lemma A.1.** *Let  $Q$  and  $Q^*$  be compact, connected, orientable, irreducible,  $\partial$ -irreducible 3-manifolds. Suppose  $Q$  is obtained by Dehn filling along some of the boundary components of  $Q^*$ . Then  $h(Q) \leq h(Q^*)$ .*

*Proof.* Let  $V = V_1 \cup \dots \cup V_p$  be the union of the solid tori attached to  $Q^*$  in order to get  $Q$ . Let  $S = S_1 \cup \dots \cup S_{h(Q)}$  be a Haken system for  $Q$ . Isotop  $S$  so that it meets  $V$  in a collection of meridinal disks and the number of such disks is minimal. Let  $S^* = S \cap Q^*$ . Then  $S^*$  is properly embedded in  $Q^*$  and has  $h(Q)$  components. It suffices to show that they are incompressible,  $\partial$ -incompressible, and pairwise non-parallel in  $Q^*$ .

*$S^*$  is incompressible in  $Q^*$ :* Suppose  $D^*$  is a compressing disk for  $S^*$  in  $Q^*$ . Then  $\partial D^* = \partial D$  for a disk  $D$  in  $S$ , and  $D \cup D^*$  bounds a 3-ball  $B$  in  $Q$ . Isotoping  $D$  across  $B$  and off  $D^*$  reduces the number of components of  $S \cap V$ , contradicting minimality.

*$S^*$  is  $\partial$ -incompressible in  $Q^*$ :* Suppose  $\Delta$  is a  $\partial$ -compressing bigon for  $S^*$  in  $Q^*$ . Then  $\partial \Delta = \alpha \cup \beta$ , where  $\alpha$  is a properly embedded arc in  $S^*$  and  $\beta$  is a properly embedded spanning arc in an annulus  $A$  in  $\partial V$  such that  $\partial A = \partial D_0 \cup \partial D_1$ , where  $D_0$  and  $D_1$  are components of  $S \cap V$ . Let  $E$  be the 3-ball in  $V$  bounded by  $A \cup D_0 \cup D_1$ . A regular neighborhood of  $\Delta \cup E$  in  $Q$  is a 3-ball across which one can isotop  $S$  to remove  $D_0$  and  $D_1$  from  $S \cap V$ , again contradicting minimality. (Alternatively, one can show from this configuration that  $S^*$  is compressible in  $Q^*$ .)

*The components of  $S^*$  are pairwise non-parallel in  $Q^*$ :* Suppose components  $S_0^*$  and  $S_1^*$  are parallel in  $Q^*$ . These surfaces are the intersections with  $Q^*$  of components  $S_0$  and  $S_1$  of  $S$ . There is an embedding of  $W^* = S_0^* \times [0, 1]$  in  $Q^*$  with  $S_0^* = S_0^* \times \{0\}$ ,

$S_1^* = S_0^* \times \{0\}$ , and  $\partial S_0^* \times [0, 1]$  contained in  $\partial Q^*$ . Each component  $A$  of  $W^* \cap \partial V$  is an annulus for which there exists a 3-ball  $B$  in  $V$  with  $A = B \cap W^*$  such that the closure of  $\partial B - A$  consists of components of  $S \cap V$ . These  $B$  allow one to extend the product structure on  $W^*$  to a product structure  $W = S_0 \times [0, 1]$  in  $Q$  with  $S_0 \times \{0\} = S_0$ ,  $S_0 \times \{1\} = S_1$ , and  $\partial S_0 \times [0, 1]$  contained in  $\partial Q$ , contradicting the fact that  $S$  is a Haken system in  $Q$ .  $\square$

**Corollary A.1.** *Let  $Y_n$  be a sequence of compact, connected, orientable 3-manifolds such that the complement  $U_n$  of the torus boundary components of  $Y_n$  is a finite volume hyperbolic manifold with totally geodesic boundary.*

$$\text{If } \lim_{n \rightarrow \infty} h(Y_n) = \infty, \text{ then } \lim_{n \rightarrow \infty} \text{Vol}(U_n) = \infty.$$

*Proof.* Let  $M_n$  be the double of  $Y_n$  along the union  $F$  of its non-torus boundary components. Suppose  $S$  is a Haken system in  $Y_n$ . Let  $\hat{S}$  be the double of  $S$  along  $S \cap F$ . The incompressibility of  $\hat{S}$  in  $M_n$  follows from the incompressibility of  $F$  and  $S$  in  $Y_n$  and the  $\partial$ -incompressibility of  $S$  in  $Y_n$ . The fact that  $\partial M_n$  consists of tori then implies that  $\hat{S}$  is  $\partial$ -incompressible in  $M_n$ . Suppose two components  $\hat{S}_0$  and  $\hat{S}_1$  of  $\hat{S}$  are parallel in  $M_n$  via a product  $\hat{S}_0 \times [0, 1]$ . This product is invariant under the involution which interchanges the two copies of  $Y_n$ . By [?, Theorem A] the restriction of the involution is equivalent to a product involution. Hence the fixed point set consists of annuli. By [30, Lemma 3.4] they are isotopic to product annuli. It follows that  $S_0$  and  $S_1$  are parallel in  $Y_n$ . Thus  $h(M_n) \geq h(Y_n)$ . The result then follows from the Theorem and the fact that the interior  $N_n$  of  $M_n$  is a complete, finite volume hyperbolic 3-manifold with  $\text{Vol}(N_n) = 2\text{Vol}(U_n)$ .  $\square$

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